HOSSAM GHANEM

(35) 11.1 11.2 Calculus with parametric curves

Example 1 30 January 2008

Let C be the parametric curve

$$x = \cos^3 t \quad , \quad y = \sin^3 t \quad , \quad 0 \le t \le \frac{\pi}{2}$$

(a) Find the point on C where the tangent line is parallel to the line $y = -\sqrt{3}\, x$.
(b) Find the length of C.

Solution

$$x = \cos^3 t \qquad\qquad y = \sin^3 t$$

$$\frac{dx}{dt} = -3\cos^2 t\, \sin t \qquad\qquad \frac{dy}{dt} = 3\sin^2 t \cos t$$

(a)

$$m = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3\sin^2 t \cos t}{-3\cos^2 t\, \sin t} = -\tan t$$

$$y = -\sqrt{3}\, x \quad \rightarrow \quad m = -\sqrt{3}$$

$$-\tan t = -\sqrt{3}$$

$$\tan t = \sqrt{3}$$

$$t = \frac{\pi}{3}$$

$$x|_{t=\frac{\pi}{3}} = \cos^3 \frac{\pi}{3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$y|_{t=\frac{\pi}{3}} = \sin^3 \frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$$

$$\therefore p\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$$

(b)

$$\left(\frac{dx}{dt}\right)^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = 9\sin^4 t \cos^2 t$$

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = 9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t = 9\sin^2 t \cos^2 t\,(\sin^2 t + \cos^2 t) = 9\sin^2 t \cos^2 t$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = 3\int_0^{\frac{\pi}{2}} \sin t \cos t \;\; dt = 3 \cdot \frac{1}{2}\Big[\sin^2 t\Big]_0^{\frac{\pi}{2}} = \frac{3}{2}(1-0) = \frac{3}{2}$$

Example 2

32 January 2009 A

A curve C has parameterization $x = 2t^3 - 6t$; $y = 2t^3 + 3t^2$ where $-1 \le t \le 1$.
Find the coordinates of the points on C at which the tangent line has slope $1/3$.

Solution

$$x = 2t^3 - 6t \qquad y = 2t^3 + 3t^2$$

$$\frac{dx}{dt} = 6t^2 - 6 \qquad \frac{dy}{dt} = 6t^2 + 6t$$

$$m = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6t^2 + 6t}{6t^2 - 6}$$

$$m = \frac{dy}{dt} = \frac{1}{3}$$

$$\frac{6t^2 + 6t}{6t^2 - 6} = \frac{1}{3}$$

$$18t^2 + 18t = 6t^2 - 6$$

$$12t^2 + 18t + 6 = 0$$

$$2t^2 + 3t + 1 = 0$$

$$(2t + 1)(t + 1) = 0$$

$$t = \frac{-1}{2} \quad \text{or} \quad t = -1$$

$$x|_{t=\frac{-1}{2}} = 2\left(\frac{-1}{2}\right)^3 - 6\left(\frac{-1}{2}\right) = \frac{-1}{4} + 3 = \frac{-1 + 12}{4} = \frac{11}{4}$$

$$y|_{t=\frac{-1}{2}} = 2\left(\frac{-1}{2}\right)^3 + 3\left(\frac{-1}{2}\right)^2 = \frac{-1}{4} + \frac{3}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\therefore p\left(\frac{11}{4}, \frac{1}{2}\right)$$

$$x|_{t=-1} = 2(-1)^3 - 6(-1) = -2 + 6 = 4$$

$$y|_{t=-1} = 2(-1)^3 + 3(-1)^2 = -2 + 3 = 1$$

$$\therefore p(4,1)$$

Example 3 9 June 1997

If the curve C is given parametrically as

$$x(t) = \ln(1-t)\,,\; y(t) = 2\sin^{-1}\sqrt{t}\;,\quad \frac{1}{9} \le t \le \frac{1}{4}$$ then find the length of C

Solution

$x = \ln(1-t)$

$$\frac{dx}{dt} = \frac{-1}{1-t}$$

$y = 2\sin^{-1}\sqrt{t}$

$$\frac{dy}{dt} = 2\cdot\frac{1}{2\sqrt{t}}\cdot\frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{t}\sqrt{1-t}}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{(1-t)^2}$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{-1}{t(1-t)}$$

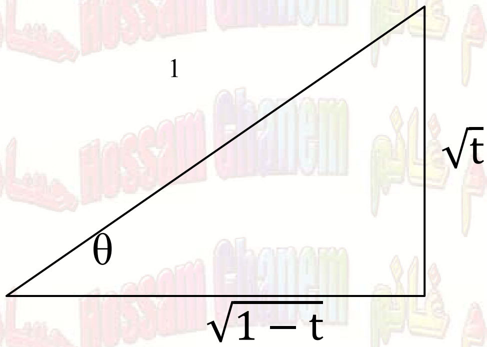
$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 = \frac{1}{t(1-t)} + \frac{1}{(1-t)^2} = \frac{1-t+t}{t(1-t)^2} = \frac{1}{t(1-t)^2}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\; dt$$

$$L = \int_{\frac{1}{9}}^{1/4} \frac{1}{\sqrt{t}(1-t)}\; dt$$

$$\sqrt{t} = \sin\theta \qquad \frac{1}{2\sqrt{t}}\,dt = \cos\theta\; d\theta$$

$$\sin\theta = \frac{\sqrt{t}}{1}$$



$$\theta = \sin^{-1}\sqrt{t}$$

$$I = \int \frac{1}{1-\sin^2\theta}\cdot 2\cos\theta\; d\theta$$

$$= \int \frac{1}{\cos^2\theta}\cdot 2\cos\theta\; d\theta = 2\int \frac{1}{\cos\theta}\, d\theta$$

$$= 2\int \sec\theta\; d\theta = 2\ln|\sec\theta + \tan\theta| + c$$

$$= 2\ln\left(\frac{1}{\sqrt{1-t}} + \frac{\sqrt{t}}{\sqrt{1-t}}\right) + c = 2\ln\left(\frac{1+\sqrt{t}}{\sqrt{1-t}}\right) + c = \ln\left(\frac{1+\sqrt{t}}{\sqrt{1-t}}\right)^2 + c = \ln\left(\frac{1+t+2\sqrt{t}}{1-t}\right) + c$$

$$L = \left[\ln\left(\frac{1+t+2\sqrt{t}}{1-t}\right)\right]_{\frac{1}{9}}^{\frac{1}{4}}$$

$$= \ln\frac{1+\frac{1}{4}+\frac{2}{2}}{\frac{3}{4}} - \ln\frac{1+\frac{1}{9}+\frac{2}{3}}{\frac{8}{9}}$$

$$= \ln\frac{4+1+4}{3} - \ln\frac{9+1+6}{8} = \ln\frac{9}{3} - \ln\frac{16}{8} = \ln 3 - \ln 2 = \ln\frac{3}{2}$$